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T - ROUGH FUZZY SET ON THE FUZZY APPROXIMATION SPACES

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ABSTRACT

Fuzzy set theory and rough set theory have many applications in the fields of data mining, knowledge representation. Nowadays, there are many extensions which are mentioned along with the properties and applications of them. Concept T- rough set which allows us to discover knowledge is expressed as a map. In this paper, we introduce the concept of T- rough fuzzy set on crisp approximation spaces; their properties and fuzzy topology spaces which based on definable sets are studied. Then, by the same way, we also introduced the concept of collective T - rough fuzzy fuzzy approximation space, it is seen as a more general concept of T- rough fuzzy set on crisp approximation spaces.

Keywords: Fuzzy approximation spaces, fuzzy topology spaces, rough fuzzy set.

Tập T- mờ thô trên không gian xấp xỉ mờ

TÓM TẮT

Lý thuyết tập hợp mờ và lý thuyết tập thô có nhiều ứng dụng trong các lĩnh vực khai thác dữ liệu, biểu diễn tri thức. Ngày nay, có rất nhiều phần mở rộng được đề cập cùng với các thuộc tính và các ứng dụng của họ. Khái niệm T- tập thô cho phép chúng ta khám phá kiến thức được thể hiện như một ánh xạ. Trong bài báo này chúng tôi đưa ra các khái niệm về tập mờ T- thô trên không gian xấp xỉ rõ; các tính chất của chúng và không gian tô pô mờ dựa trên bộ định nghĩa được nghiên cứu. Sau đó, bằng cách thức tương tự, chúng tôi cũng giới thiệu khái niệm tập T – mờ thô trên không gian xấp xỉ mờ được xem như là một kết quả khái quát hơn kết quả về tập mờ T - thô trên không gian xấp xỉ rõ.

Từ khóa: Không gian tô pô mờ, không gian xấp xỉ mờ, T- mờ thô.

1. INTRODUCTION

Rough set theory was introduced by Pawlak in the 1980s. It has become a useful mathematical tool for data mining, especially for redundant and uncertain data. At first, the establishment of rough set theory was based on equivalence relation. The set of equivalence classes of the universal set, obtained by an equivalence relation, was the basis for the construction of the upper and lower approximations of the subset of the universal set. Typical applications of rough set theory were to find the attribute reductions in information systems and decision systems. Equivalence relation was often used to the indiscernibility relation. Over time, the application of rough set theory in data mining became increasingly diverse. The demand for an equivalence relation on the universal set seemed to be too strict requirements (Dubois (1990), Yao (1997), Kryszkiewicz (1999),...). It was a more limited application of rough set theory in data mining. For example, real-valued information systems (Kryszkiewicz (1999)) and incomplete information systems (Hu *et al.* (2006)) cannot be handled with Pawlak's rough sets. Because of this limitation, nowadays, there are many extensions of rough set theory. Bonikowski *et al.* (1998) introduced the concept of rough sets with covering. Yao (1998) introduced the concept of rough sets based on relations. In 2008, Davvar also studied the concept of generalized rough sets and called them T-rough sets. Ali *et al.* (2013) investigated some properties of T-rough sets. Besides rough theory, fuzzy set theory, which was introduced by Zadeh (1965), is also a useful mathematical tool for processing uncertainty and incomplete information for the information systems. In addition, combining rough sets and fuzzy sets also has many interesting results. The approximation of rough sets (or fuzzy sets) in fuzzy approximation spaces gives us the fuzzy rough sets (Yao (1997), Dubois (1990)) and the approximation of fuzzy sets in crisp approximation spaces gives us the rough fuzzy sets (Yao (1997), Wu *et al.* (2003, 2013), Du (2012), Dubois (1990), Tan *et al.* (2015)). Wu *et al.* (2003) presented a general framework for the study of fuzzy rough sets in both constructive and axiomatic approaches. By the same, Wu (2013) and Xu (2009) investigated the fuzzy topology structures on rough fuzzy sets, in which both constructive and axiomatic approaches were used. The concept of the rough fuzzy set on approximation space was introduced by Banerjee and Pal (1996). Later, Liu (2004) extended this rough fuzzy set on the fuzzy approximation spaces. Zhao *et al.* (2009) extended Liu's results by defining the lower and upper approximations. It is well-known that the rough set theory is closely related to the topology theory (Wu (2013), Hu (2014),...). A natural question: are similar results as above true when we combine T- rough sets and fuzzy sets, in which, $T: X \rightarrow P^*(Y)$ is a (crisp) set-value map? It is well-know that crisp set is a specific case of fuzzy sets, so we can build the similar results when replacing $T: X \rightarrow P^*(Y)$ (a crisp set-value map) with $T: X \rightarrow \mathcal{F}(Y)$ (a fuzzy set-value map). In this paper, we provide a few answers to these situations.

The remaining part of this paper is organized as follows: In section 2, we quote some definitions of T-rough sets and α-cut of a fuzzy set. In section 3, we define the T-rough fuzzy sets along with upper and lower rough fuzzy approximation operators on crisp approximation spaces and their properties. In section 4, we study the fuzzy topological structures associated with definable sets. Finally, we introduce T-rough fuzzy sets along with upper and lower rough fuzzy set approximation operators on fuzzy approximation spaces and their properties in section 5.

2. T-ROUGH SETS AND FUZZY SETS

Definition 2.1. [1]. Let X and Y be two nonempty universes. Let T be a set-value mapped by $T: X \rightarrow P^*(Y)$, where $P^*(Y)$ is the collection of all (non empty) subsets of Y. Then the (X, Y, T) is referred to as a generalized (crisp) approximation space. For any subset $A \in P^*(Y)$, a pair of lower and upper approximations of A, $\underline{T}(A)$ A, and $\bar{T}(A)$, are subsets of X which are defined respectively by

$$\underline{T}(A) = \{ x \in X: T(x) \subset A\}$$

and

$$\bar{T}(A) = \{ x \in X: T(x) \cap A \neq \emptyset\}$$

The pair $(\underline{T}(A), \bar{T}(A))$ is called a generalized rough set (T- rough set).

Note that $\underline{T}$, $\bar{T}$ $: P^*(Y) \rightarrow P^*(X)$ are the lower and upper generalized rough approximation operators, respectively.

We denote $\mathcal{F}(Y)$ is the collection fuzzy subsets of Y. Then for all $A \in \mathcal{F}(Y)$, we use $A(y)$ to denote the grade of membership of y in A.

Definition 2.2. For $A \in \mathcal{F}(Y)$ and for all $\alpha \in [0,1]$, the α −cut and strong α −cut of fuzzy set A, denoted A_α and A_α^s respectively, are defined as follows:

$$A_\alpha = \{y \in Y: A(y) \geq \alpha\}, \qquad A_\alpha^s = \{y \in Y: A(y) > \alpha\}.$$

Theorem 2.1. [13]. Let $A \in \mathcal{F}(Y)$, then $A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha$ and $A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha^s$, and for all $y \in Y$, we have $A(y) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge A_\alpha(y)) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge A_\alpha^s(y))$.

3. T-ROUGH FUZZY SET AND ITS PROPERTIES

Definition 3.1. Let (X, Y, T) be a generalized (crisp) approximation space. For all $A \in \mathcal{F}(Y)$, we define:

$\underline{T}(A)_\propto = \{x \in X: T(x) \subset A_\alpha\}$,

$\overline{T}(A)_\propto = \{x \in X: T(x) \cap A_\alpha \neq \emptyset\}$

and

$\underline{T}(A)^s_\alpha = \{x \in X: T(x) \subset A^s_\alpha\}$,

$\overline{T}(A)^s_\propto = \{x \in X: T(x) \cap A^s_\alpha \neq \emptyset\}$

Example 3.1. Given $= \{a, b, c\}$, $Y = \{1, 2, 3, 4\}$. Let T be a set-value mapped by $T: X \to P^*(Y)$, where $T(a) = \{1,2,4\}, T(b) = \{1,3,4\}, T(c) = \{2,3\}$. For a fuzzy subset $A = \frac{0.3}{1} + \frac{0.4}{2} + \frac{0.5}{3} + \frac{0.8}{4}$ of Y we have

$\underline{T}(A)_{0.4} = \{x \in X: T(x) \subset A_{0.4}\} = \{c\}$;

$\overline{T}(A)_{0.4} = \{x \in X: T(x) \cap A_{0.4} \neq \emptyset\} = \{a, b, c\} = X$.

$\underline{T}(A)^s_{0.4} = \{x \in X: T(x) \subset A^s_{0.4}\} = \emptyset$;

$\overline{T}(A)^s_{0.4} = \{x \in X: T(x) \cap A^s_{0.4} \neq \emptyset\} = \{a, b, c\} = X$.

Lemma 3.1. Let (X, Y, T) be a generalized (crisp) approximation space. For all $A, B \in \mathcal{F}(Y), \forall \propto \in [0,1]$, we have

$\underline{T}(A \cap B)_\propto = \underline{T}(A)_\propto \cap \underline{T}(B)_\propto$

$\overline{T}(A \cap B)_\propto \subseteq \overline{T}(A)_\propto \cap \overline{T}(B)_\propto$;

$\underline{T}(A \cap B)^s_\alpha = \underline{T}(A)^s_\alpha \cap \underline{T}(B)^s_\alpha$

$\overline{T}(A \cap B)^s_\propto \subseteq \overline{T}(A)^s_\propto \cap \overline{T}(B)^s_\propto$;

$\underline{T}(A)_\propto \cup \underline{T}(B)_\propto \subset \underline{T}(A \cup B)_\propto$;

$\overline{T}(A \cup B)_\propto = \overline{T}(A)_\propto \cup \overline{T}(B)_\propto$;

$\underline{T}(A)^s_\alpha \cup \underline{T}(B)^s_\alpha \subseteq \underline{T}(A \cup B)^s_\alpha$

$\overline{T}(A \cup B)^s_\propto = \overline{T}(A)^s_\propto \cup \overline{T}(B)^s_\propto$

If $A \subset B$ then $\underline{T}(A)_\propto \subseteq \underline{T}(B)_\propto$ and $\overline{T}(A)_\propto \subseteq \overline{T}(B)_\propto$;

If $A \subset B$ then $\underline{T}(A)^s_\alpha \subseteq \underline{T}(B)^s_\alpha$ and $\overline{T}(A)^s_\propto \subseteq \overline{T}(B)^s_\propto$;

$\underline{T}(A^C)_\propto = (\overline{T}(A^s_{1-\propto}))^C \quad ; \overline{T}(A^C)_\propto = \left(\underline{T}(A^s_{1-\propto})\right)^C$

Theorem 3.1. Let (X, Y, T) be a generalized (crisp) approximation space. For all $A \in \mathcal{F}(Y)$, we have

$(\cup_{\alpha \in [0,1]} \alpha \underline{T}(A)_\alpha)(x)$

$= \vee_{\alpha \in [0,1]}\{\propto: \forall y \in T(x) \wedge A(y) \geq \propto\}$

$(\cup_{\alpha \in [0,1]} \alpha \underline{T}(A)^s_\propto)(x)$

$= \vee_{\alpha \in [0,1]}\{\propto: \forall y \in T(x) \wedge A(y) > \propto\}$

$(\cup_{\alpha \in [0,1]} \alpha \overline{T}(A)_\alpha)(x)$

$= \vee_{\alpha \in [0,1]}\{\propto: \exists y \in T(x) \wedge A(y) \geq \propto\}$

$(\cup_{\alpha \in [0,1]} \alpha \overline{T}(A)^s_\propto)(x)$

$= \vee_{\alpha \in [0,1]}\{\propto: \exists y \in T(x) \wedge A(y) > \propto\}$.

Proof.

$(\cup_{\alpha \in [0,1]} \alpha \underline{T}(A)_\alpha)(x)$

$= \vee_{\alpha \in [0,1]}\{\propto \wedge \underline{T}(A)_\alpha(x)\}$

$= \vee_{\alpha \in [0,1]}\{\propto: T(x) \subset A_\alpha\}$

$= \bigvee_{\alpha \in [0,1]} \{\propto: \forall y \in T(x) \; and \; A(y) \geq \propto\}$

$(\cup_{\alpha \in [0,1]} \alpha \underline{T}(A)^s_\propto)(x)$

$= \vee_{\alpha \in [0,1]}\{\propto \wedge \underline{T}(A)^s_\alpha(x)\}$

$= \vee_{\alpha \in [0,1]}\{\propto: \forall y \in T(x) \wedge A(y) > \propto\}$

$(\cup_{\alpha \in [0,1]} \alpha \overline{T}(A)_\alpha)(x)$

$= \bigvee_{\alpha \in [0,1]} \left\{\propto \bigwedge \overline{T}(A)_\alpha(x)\right\}$

$= \vee_{\alpha \in [0,1]}\{\propto: T(x) \cap A_\alpha \neq \emptyset\}$

$= \vee_{\alpha \in [0,1]}\{\propto: \exists y \in T(x) \wedge A(y) \geq \propto\}$

$(\cup_{\alpha \in [0,1]} \alpha \overline{T}(A)^s_\propto)(x)$

$= \bigvee_{\alpha \in [0,1]} \left\{\propto \bigwedge \overline{T}(A)^s_\alpha(x)\right\}$

$= \bigvee_{\alpha \in [0,1]} \{\propto: T(x) \cap A^s_\propto \neq \emptyset\}$

$= \vee_{\alpha \in [0,1]}\{\propto: \exists y \in T(x) \wedge A(y) > \propto\}$

Definition 3.2. Let (X, Y, T) be a generalized (crisp) approximation space. For all $A \in \mathcal{F}(Y)$, we define:

$\underline{T}(A) = \cup_{\alpha \in [0,1]} \alpha \underline{T}(A)_\alpha$

$\overline{T}(A) = \cup_{\alpha \in [0,1]} \alpha \overline{T}(A)_\alpha$

The pair $(\underline{T}(A), \overline{T}(A))$ is called a generalized rough fuzzy set (T-rough fuzzy set).

Note that $\underline{T}, \overline{T}: \mathcal{F}(Y) \to \mathcal{F}(X)$ are the lower and upper generalized rough fuzzy approximation operators, respectively.

Example 3.2. Let T be a set-value mapped by $T: X \to P^*(Y)$ which was defined in example 3.1 and a fuzzy subset $A = \frac{0.3}{1} + \frac{0.4}{2} + \frac{0.5}{3} + \frac{0.8}{4}$ in Y. We easily verify that

$\underline{T}(A)_{0.2} = \{a, b, c\} = X = \underline{T}(A)_{0.3}$

$\underline{T}(A)_{\propto} = \{a, b, c\}, \forall \propto \leq 0.3;$

$\underline{T}(A)_{\propto} = \underline{T}(A)_{0.4} = \{c\}, \forall \propto \in (0.3, 0.4]$

$\underline{T}(A)_{\propto} = \emptyset, \forall \propto > 0.4.$

Then $\underline{T}(A) = \cup_{\alpha \in [0,1]} \alpha \underline{T}(A)_\alpha = \frac{0.3}{a} + \frac{0.3}{b} + \frac{0.4}{c}$.
Similarly $\overline{T}(A) = \cup_{\alpha \in [0,1]} \alpha \overline{T}(A)_\alpha = \frac{0.8}{a} + \frac{0.8}{b} + \frac{0.5}{c}$.

Now we consider some properties of T-rough fuzzy sets.

Theorem 3.2. Let (X, Y, T) be a generalized (crisp) approximation space. Then, its lower and upper generalized rough fuzzy approximation operators satisfy the following. For all $A, B \in \mathcal{F}(Y)$

(L1) $\underline{T}(A) = (\overline{T}(A^C))^C$

(L2) $\underline{T}(Y) = X$

(L3) $\underline{T}(A \cap B) = \underline{T}(A) \cap \underline{T}(B)$

(L4) $A \subseteq B \Rightarrow \underline{T}(A) \subseteq \underline{T}(B)$

(L5) $\underline{T}(A) \cup \underline{T}(B) \subseteq \underline{T}(A \cup B)$

(L6) $\underline{T}(A \cup \hat{\propto}) = \underline{T}(A) \cup \hat{\propto}$

(U1) $\overline{T}(A) = (\underline{T}(A^C))^C$

(U2) $\overline{T}(\emptyset) = \emptyset$

(U3) $\overline{T}(A \cup B) = \overline{T}(A) \cup \overline{T}(B)$

(U4) $A \subseteq B \Rightarrow \overline{T}(A) \subseteq \overline{T}(B)$

(U5) $\overline{T}(A \cap B) \subseteq \overline{T}(A) \cap \overline{T}(B)$

(U6) $\overline{T}(A \cap \hat{\propto}) = \overline{T}(A) \cap \hat{\propto}$

where
$\forall \propto \in [0,1], \forall y \in Y, \forall x \in X: \hat{\propto}(x) = \hat{\propto}(y) = \propto$.

4. FUZZY TOPOLOGICAL SPACES

Let (X, Y, T) be a generalized (crisp) approximation space.

Lemma 4.1. Let X and Y be two nonempty universes. Let $T: X \to P^*(Y)$ be a set-valued map. Then $\underline{T}(Y) = X = \overline{T}(Y)$ and $\underline{T}(\emptyset) = \emptyset = \overline{T}(\emptyset)$.

Lemma 4.2. Let X and Y be two nonempty universes. Let $T: X \to P^*(Y)$ be a set-valued map. Then $\underline{T}(A) \subseteq \overline{T}(A)$ for all $A \in \mathcal{F}(Y)$.

Definition 4.1. Let X and Y be two nonempty universes. Let T be a set-value mapped by $T: X \to P(Y)$. We define binary relation $\eqsim, \simeq, \sim$ on $P^*(Y)$ by defining:

$A \eqsim B \Leftrightarrow \overline{T}(A)_\propto = \overline{T}(B)_\propto, \forall \propto \in [0,1]$

$A \simeq B \Leftrightarrow \underline{T}(A)_\propto = \underline{T}(B)_\propto, \forall \propto \in [0,1]$

$A \sim B \Leftrightarrow \overline{T}(A)_\propto = \overline{T}(B)_\propto$

and $\underline{T}(A)_\propto = \underline{T}(B)_\propto, \forall \propto \in [0,1]$.

It is easy to verify that $\eqsim, \simeq, \sim$ are equivalence relations on $P^*(Y)$ and called the generalized rough fuzzy upper equal relation, generalized rough fuzzy lower equal relation, and generalized rough fuzzy equal relation, respectively.

Theorem 4.1. Let (X, Y, T) be a generalized (crisp) approximation. Then for all $A, B, A_1, B_1 \in \mathcal{F}(Y)$, we have

$A \eqsim B \Leftrightarrow A \eqsim (A \cup B) \eqsim B$

$A \eqsim A_1, B \eqsim B_1 \Rightarrow (A \cup B) \eqsim (A_1 \cup B_1)$

$A \eqsim B \Rightarrow (A \cup B^C) \eqsim Y$

$A \subseteq B, B \eqsim \emptyset \Rightarrow A \eqsim \emptyset$

$A \subseteq B, A \eqsim Y \Rightarrow B \eqsim Y$

Similarly, we have

Theorem 4.2. Let (X, Y, T) be a generalized (crisp) approximation. Then for all $A, B, A_1, B_1 \in \mathcal{F}(Y)$, we have

$A \simeq B \Leftrightarrow A \simeq (A \cup B) \simeq B$

$A \simeq A_1, B \simeq B_1 \Rightarrow (A \cup B) \simeq (A_1 \cup B_1)$

$A \simeq B \Rightarrow (A \cap B^C) \simeq \emptyset.$

$A \subseteq B, B \simeq \emptyset \Rightarrow A \simeq \emptyset$

$A \subseteq B, A \simeq Y \Rightarrow B \simeq Y$

Now, we introduce the definition related to fuzzy topology (Lowen (1976)).

Definition 4.2. A collection of subsets τ of $\mathcal{F}(Y)$ is referred to as a fuzzy topology on Y if it satisfies:

If $\mathcal{A} \subseteq \tau$ then $\cup_{A_i \in \mathcal{A}} A_i \in \tau$

If $A, B \in \tau$ then $A \cap B \in \tau$.

If $\hat{\propto} \in \mathcal{F}(Y)$ then $\hat{\propto} \in \tau$.

Let X and Y be two nonempty universes. Let $T: X \to P^*(Y)$ be a set-valued map. We denote $\tau = \{A \in \mathcal{F}(Y): \underline{T}(A) = \overline{T}(A)\}$. Then, we have some properties as follows:

Theorem 4.3. Let X and Y be two nonempty universes. Let $T: X \to P^*(Y)$ be a set-valued map. (Y, τ) is a fuzzy topological space.

Proof. Lemma 4.1 shows that $\emptyset, Y \in \tau$. We consider all the conditions of definition 4.2 for this space.

Given $\mathcal{A} = \{A_i\} \subseteq \tau$, then

$\underline{T}(A_i)_\propto = \overline{T}(A_i)_\propto \ \forall \propto \in [0,1]$.

Since $\overline{T}(\cup_{A_i \in \mathcal{A}} A_i)_\propto = \cup_{A_i \in \mathcal{A}} \overline{T}(A_i)_\propto$

$= \cup_{A_i \in \mathcal{A}} \underline{T}(A_i)_\propto \subseteq \underline{T}(\cup_{A_i \in \mathcal{A}} A_i)_\propto, \forall \propto \in [0,1]$

And $\underline{T}(\cup_{A_i \in \mathcal{A}} A_i)_\propto \subseteq \overline{T}(\cup_{A_i \in \mathcal{A}} A_i)_\propto$ (lemma 4.2)

then $\overline{T}(\cup_{A_i \in \mathcal{A}} A_i)_\propto = \underline{T}(\cup_{A_i \in \mathcal{A}} A_i)_\propto, \forall \propto \in [0,1]$. So $\cup_{A_i \in \mathcal{A}} A_i \in \tau$.

If $A, B \in \tau$ then $\underline{T}(A) = \overline{T}(A), \underline{T}(B) = \overline{T}(B)$. So that $\overline{T}(A \cap B) \subseteq \overline{T}(A) \cap \overline{T}(B) = \underline{T}(A \cap B)$. Application of Lemma 3.2, we have $A \cap B \in \tau$.

For any $\hat{\propto} \in \mathcal{F}(Y)$. It is easy to see that $\underline{T}(\hat{\propto})(x) = \wedge_{y \in T(x)} \hat{\propto}(y) = \propto$

and $\overline{T}(\hat{\propto})(x) = \vee_{y \in T(x)} \hat{\propto}(y) = \propto$

Hence $\underline{T}(\hat{\propto}) = \overline{T}(\hat{\propto})$. So $\hat{\propto} \in \tau$.

This shows that τ is a fuzzy topology on Y. Hence (Y, τ) is a fuzzy topological space.

Similarly, we have

Theorem 4.4. Let X and Y be two nonempty universes. Let $T: X \to P^*(Y)$ be a set-valued map. Then collection

$\tau' = \{A' \in \mathcal{F}(X): \exists A \in \mathcal{F}(Y), A' = \underline{T}(A) = \overline{T}(A)\}$

is a fuzzy topology on X.

5. T-ROUGH FUZZY SETS ON FUZZY APPROXIMATION SPACES

In this section (X, Y, T) is a generalized (fuzzy) approximation space, where $T: X \to \mathcal{F}(Y)$ is a (fuzzy) set-valued map. We propose methods to build a T-rough fuzzy set in which a pair of lower and upper approximations of the fuzzy set $A \in \mathcal{F}(Y)$, $\underline{T}(A)$ and $\overline{T}(A)$, are fuzzy subsets of X. The results obtained in this section are the more general results which were obtained in section 3.

Definition 5.1. Let (X, Y, T) be a generalized (fuzzy) approximation space. For all $A \in \mathcal{F}(Y)$, we define:

$\underline{T}(A)_\propto = \{x \in X: T(x)_\propto \subseteq A_\alpha\}$

$\overline{T}(A)_\propto = \{x \in X: T(x)_\propto \cap A_\alpha \neq \emptyset\}$

$\underline{T}(A)^s_\alpha = \{x \in X: T(x)^s_\propto \subseteq A^s_\alpha\}$

$\overline{T}(A)^s_\propto = \{x \in X: T(x)^s_\propto \cap A^s_\alpha \neq \emptyset\}$

Example 5.1. Given $= \{a, b, c\}$, $Y = \{1, 2, 3, 4\}$. Let T be a set-valued map by $T: X \to \mathcal{F}(Y)$ and

$$T(a) = \frac{0.1}{1} + \frac{0.2}{2} + \frac{0.5}{3} + \frac{0.9}{4}$$

$$T(b) = \frac{0.8}{1} + \frac{0.6}{2} + \frac{0.9}{3} + \frac{0.8}{4}$$

$$T(c) = \frac{0.6}{1} + \frac{0.9}{2} + \frac{0.8}{3} + \frac{0.7}{4}.$$

For a fuzzy subset $A = \frac{0.3}{1} + \frac{0.4}{2} + \frac{0.5}{3} + \frac{0.8}{4}$ of Y we have $\underline{T}(A)_{0.7} = \{x \in X: T(x)_{0.7} \subset A_{0.7}\} = \{a\}$;

$\overline{T}(A)_{0.7} = \{x \in X: T(x)_{0.7} \cap A_{0.7} \neq \emptyset\} = \{a, b, c\} = X$.

$\underline{T}(A)^s_{0.7} = \{x \in X: \underline{T}(x)^s_{0.7} \subseteq A^s_{0.7}\} = \{a\}$;

$\overline{T}(A)^s_{0.7} = \{x \in X: \underline{T}(x)^s_{0.7} \cap A^s_{0.7} \neq \emptyset\} = \{a, b\}$.

Lemma 5.1. Let (X, Y, T) be a generalized (fuzzy) approximation space. For all $A, B \in \mathcal{F}(Y), \forall \propto \in [0,1]$, we have

$\underline{T}(A \cap B)_\propto = \underline{T}(A)_\propto \cap \underline{T}(B)_\propto$,

$\overline{T}(A \cap B)_\propto \subseteq \overline{T}(A)_\propto \cap \overline{T}(B)_\propto$

$\underline{T}(A \cap B)^s_\alpha = \underline{T}(A)^s_\alpha \cap \underline{T}(B)^s_\alpha$

$\overline{T}(A \cap B)^s_\propto \subseteq \overline{T}(A)^s_\propto \cap \overline{T}(B)^s_\propto$

$\underline{T}(A)_\propto \cup \underline{T}(B)_\propto \subset \underline{T}(A \cup B)_\propto$

$\overline{T}(A \cup B)_\propto = \overline{T}(A)_\propto \cup \overline{T}(B)_\propto$

$\underline{T}(A)^s_\alpha \cup \underline{T}(B)^s_\alpha \subseteq \underline{T}(A \cup B)^s_\alpha$

$\overline{T}(A \cup B)^s_\propto = \overline{T}(A)^s_\propto \cup \overline{T}(B)^s_\propto$;

If $A \subset B$ then $\underline{T}(A)_\propto \subseteq \underline{T}(B)_\propto$ and $\overline{T}(A)_\propto \subseteq \overline{T}(B)_\propto$

If $A \subset B$ then $\underline{T}(A)^s_\alpha \subseteq \underline{T}(B)^s_\alpha$ and $\overline{T}(A)^s_\propto \subseteq \overline{T}(B)^s_\propto$

$\underline{T}(A^C)_\propto = (\overline{T}(A^s_{1-\propto}))^C$

$\overline{T}(A^C)_\propto = \left(\underline{T}(A^s_{1-\propto})\right)^C$

Proof.

(1). We have

$\underline{T}(A \cap B)_{\propto} = \{x \in X : T(x)_{\propto} \subseteq (A \cap B)_{\propto}\}$

$= \{x \in X : T(x)_{\propto} \subseteq A_{\propto} \cap B_{\propto}\}$

$= \{x \in X : T(x)_{\propto} \subseteq A_{\propto}\} \cap \{x \in X : T(x)_{\propto} \subseteq B_{\propto}\}$

$= \underline{T}(A)_{\propto} \cap \underline{T}(B)_{\propto}$

(2). $\underline{T}(A \cap B)^s_{\alpha} = \{x \in X : T(x)^s_{\propto} \subset (A \cap B)^s_{\alpha}\}$

$= \{x \in X : T(x)^s_{\propto} \subset A^s_{\alpha} \cap B^s_{\alpha}\}$

$= \{x \in X : T(x)^s_{\propto} \subset A^s_{\alpha}\} \cap \{x \in X : T(x)^s_{\propto} \subset B^s_{\alpha}\}$

$= \underline{T}(A)^s_{\alpha} \cap \underline{T}(B)^s_{\alpha}$

(7). We have $\underline{T}(A^C)_{\propto} = \{x \in X : T(x)_{\propto} \subset A^C_{\propto}\}$

$= \left\{x \in X : T(x)_{\propto} \subset \{x \in X : \mu_{A^C}(x) \geq \propto\}\right\}$

$= \left\{x \in X : T(x)_{\propto} \subset \{x \in X : 1 - \mu_A(x) \geq \propto\}\right\}$

$= \left\{x \in X : T(x)_{\propto} \subset \{x \in X : \mu_A(x) \leq 1 - \propto\}\right\}$

$= \left\{x \in X : T(x)_{\propto} \subset \{x \in X : T(x)_{\propto} \subset \{x \in X : \mu_A\ (x) > 1 - \propto\}^C\right\}$

$= \left\{x \in X : T(x)_{\propto} \subseteq A^s_{\alpha}\ \right\}^C = (\overline{T}(A^s_{1-\propto}))^C$

The remaining properties are similarly proved.

Theorem 5.1. Let (X, Y, T) be a generalized (fuzzy) approximation space. For all $A \in \mathcal{F}(Y)$, we have

$(\bigcup_{\alpha\in[0,1]} \alpha\, \underline{T}(A)_{\alpha})(x)$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \forall y \in T(x)_{\propto}, T(x)(y) \wedge A(y) \geq \propto\}$

$(\bigcup_{\alpha\in[0,1]} \alpha\, \underline{T}(A)^s_{\propto}\ )(x)$

$\leq \bigvee_{\alpha\in[0,1]} \left\{\propto : \forall y \in T(x)_{\propto},\quad T(x)(y) \bigwedge A(y) > \propto\right\}$

$(\bigcup_{\alpha\in[0,1]} \alpha\, \overline{T}(A)_{\alpha})(x)$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \exists y \in T(x)_{\propto}, T(x)(y) \wedge A(y) \geq \propto\}$

$\leq \bigvee_{y\in Y} \left\{T(x)(y) \bigwedge A(y)\right\}$

$(\bigcup_{\alpha\in[0,1]} \alpha\, \overline{T}(A)^s_{\propto})(x)$

$= \bigvee_{\alpha\in[0,1]} \left\{\propto : \exists y \in T(x) \bigwedge A(y) > \propto\right\}$

$= \bigvee_{y\in Y} \left\{T(x)(y) \bigwedge A(y)\right\}$

Proof.

We have

$(\bigcup_{\alpha\in[0,1]} \alpha\, \underline{T}(A)_{\alpha})(x) = \bigvee_{\alpha\in[0,1]} \{\propto \wedge \underline{T}(A)_{\alpha}(x)\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : T(x)_{\propto} \subset A_{\alpha}\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \forall y \in T(x)_{\propto}\ and\ A(y) \geq \propto\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \forall y \in y, If\ T(x)(y) \geq \propto\ then\ A(y) \geq \propto\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \forall y \in T(x)_{\propto}, T(x)(y) \wedge A(y) \geq \propto\}.$

$(\bigcup_{\alpha\in[0,1]} \alpha\, \underline{T}(A)^s_{\propto})(x)$

$= \bigvee_{\alpha\in[0,1]} \left\{\propto \bigwedge \underline{T}(A)^s_{\alpha}(x)\right\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : T(x)^s_{\propto} \subset A^s_{\propto}\}$

$= \bigvee_{\alpha\in[0,1]} \left\{\propto : \forall y \in T(x)^s_{\propto},\quad T(x)(y) \bigwedge A(y) > \propto\right\}$

$\leq \bigvee_{\alpha\in[0,1]} \left\{\propto : \forall y \in T(x)_{\propto},\quad T(x)(y) \bigwedge A(y) > \propto\right\}$

$\leq \bigvee_{\alpha\in[0,1]} \left\{\propto : \forall y \in T(x)_{\propto},\quad T(x)(y) \bigwedge A(y) \geq \propto\right\}$

$(\bigcup_{\alpha\in[0,1]} \alpha\, \overline{T}(A)_{\alpha})(x)$

$= \bigvee_{\alpha\in[0,1]} \left\{\propto \bigwedge \overline{T}(A)_{\alpha}(x)\right\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : T(x)_{\propto} \cap A_{\alpha} \neq \emptyset\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \exists y \in T(x)_{\propto} \wedge\quad A_{\alpha}\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \exists y \in Y, T(x)(y) \geq \propto\ and\ A(y) \geq \propto\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \exists y \in Y, T(x)(y) \wedge A(y) \geq \propto\}$

$\leq \bigvee_{\alpha\in[0,1]} \left\{\propto : \bigvee_{y\in Y} T(x)(y) \wedge A(y) \geq \propto\right\}$

$= \bigvee_{y\in Y}(T(x)(y) \wedge A(y)).$

$(\bigcup_{\alpha\in[0,1]} \alpha\, \overline{T}(A)^s_{\propto})(x)$

$= \bigvee_{\alpha\in[0,1]} \left\{\propto \bigwedge \overline{T}(A)^s_{\alpha}(x)\right\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : T(x)^s_{\propto} \cap A^s_{\propto} \neq \emptyset\}$

$= \bigvee_{\alpha\in[0,1]} \{\propto : \exists y \in Y, (T(x)(y) \wedge A(y)) > \propto\}$

$= \bigvee_{\alpha\in[0,1]} \left\{\propto : \bigvee_{y\in Y} T(x)(y) \wedge A(y) > \propto\right\}$

$= \bigvee_{y\in Y}(T(x)(y) \wedge A(y))$

Definition 5.2. Let (X, Y, T) be a generalized (fuzzy) approximation space. For all $A \in \mathcal{F}(Y)$, we define:

$\underline{T}(A) = \bigcup_{\alpha\in[0,1]} \alpha\, \underline{T}(A)^s_{\alpha}$

$\overline{T}(A) = \cup_{\alpha\in[0,1]} \alpha\, \overline{T}(A)^{s}_{\propto}$.

The pair $(\underline{T}(A), \overline{T}(A))$ is called a generalized rough fuzzy set (T-rough fuzzy set) on the fuzzy approximation spaces.

Note that $\underline{T}$, $\overline{T}: \mathcal{F}(Y) \to \mathcal{F}(X)$ are the lower and upper generalized rough fuzzy approximation operators, respectively.

Example 5.2. Let T be a set-value mapped by $T: X \to P^*(Y)$ which was defined in example 5.1 and a fuzzy subset $A = \frac{0.3}{1} + \frac{0.4}{2} + \frac{0.5}{3} + \frac{0.8}{4}$ in Y. We easily verify that

$\underline{T}(A)_{0.2} = \{a, b, c\} = X = \underline{T}(A)_{0.3}$,

$\underline{T}(A)_{\propto} = \{a, b, c\}, \forall \propto \le 0.3$

$\underline{T}(A)_{\propto} = \underline{T}(A)_{0.8} = \{a\}, \forall \propto \in (0.3, 0.8]$

$\underline{T}(A)_{\propto} = \emptyset, \forall \propto > 0.8.$

Then

$$\underline{T}(A) = \bigcup_{\alpha\in[0,1]} \alpha\, \underline{T}(A)_{\alpha} = \frac{0.8}{a} + \frac{0.3}{b} + \frac{0.3}{c}$$

Similarly

$$\overline{T}(A) = \bigcup_{\alpha\in[0,1]} \alpha\, \overline{T}(A)_{\alpha} = \frac{0.8}{a} + \frac{0.7}{b} + \frac{0.7}{c}$$

Now, we consider some properties of T-rough fuzzy sets.

Theorem 5.2. Let (X, Y, T) be a generalized (crisp) approximation space. Then, its lower and upper generalized rough fuzzy approximation operators satisfy the following. For all $A, B \in \mathcal{F}(Y)$

(L1) $\underline{T}(A) = (\overline{T}(A^C))^C$,

(L2) $\underline{T}(Y) = X$

(L3) $\underline{T}(A \cap B) = \underline{T}(A) \cap \underline{T}(B)$

(L4) $A \subseteq B \Longrightarrow \underline{T}(A) \subseteq \underline{T}(B)$

(L5) $\underline{T}(A) \cup \underline{T}(B) \subseteq \underline{T}(A \cup B)$

(L6) $\underline{T}(A \cup \hat{\propto}) = \underline{T}(A) \cup \hat{\propto}$

(U1) $\overline{T}(A) = (\underline{T}(A^C))^C$

(U2) $\overline{T}(\emptyset) = \emptyset$

(U3) $\overline{T}(A \cup B) = \overline{T}(A) \cup \overline{T}(B)$

(U4) $A \subseteq B \Longrightarrow \overline{T}(A) \subseteq \overline{T}(B)$

(U5) $\overline{T}(A \cap B) \subseteq \overline{T}(A) \cap \overline{T}(B)$.

(U6) $\overline{T}(A \cap \hat{\propto}) = \overline{T}(A) \cap \hat{\propto}$

where

$\forall \propto \in [0,1], \forall y \in Y, \forall x \in X$: $\hat{\propto}(x) = \hat{\propto}(y) = \propto$.

Proof. These are easy obtained by using Lemma 5.1 and Definition 5.2.

According to Theorem 5.2 and by the same way in Section 4, we also obtain the fuzzy topological spaces. Let X and Y be two nonempty universes. Let $T: X \to \mathcal{F}(Y)$ be a set-valued map. We denote

$\sigma = \{A \in \mathcal{F}(Y): \underline{T}(\mathrm{A}) = \overline{T}(A)\}$

and

$\sigma' = \{A' \in X: \exists\, A \in \mathcal{F}(Y), \underline{T}(\mathrm{A}) = \overline{T}(A)\}$

Then, we have some properties as follows:

Theorem 5.3.

σ is a topology on X.

σ' is a topology on Y.

6. CONCLUSIONS

In this paper, we build the T-rough fuzzy set (definition 3.2) based on the α-cut of a fuzzy set in a crisp approximation space and study some properties of it. We also investigate fuzzy topological spaces of all the sets which were definable (theorem 4.3, theorem 4.4). Finally, we introduce the T-rough fuzzy sets (definition 5.2) based on the α-cut of a fuzzy set in the fuzzy approximation spaces. By the same way, we also point out the results obtained in the fuzzy approximation space also allow us to identify the fuzzy topological spaces (theorem 5.3), respectively.

REFERENCES

Ali M.I., Davvaz B., Shabir M. (2013). Some properties of generalized rough sets, Information Sciences, 224: 170 - 179.

Allam A. A., M.Y. Baker, E.A. Abo-Tabl (2008). Some methods for generating topologies by relations, Bulletin of the Malaysia mathematical Sciences society, 31(1): 35 - 45.

An S., Hu Q., Yu D. (2011). Case - based classifers with fuzzy rough sets, Rough set and knowledge technology (6954), pp. 172 - 177.

Bonikowski Z., E. Bryniarski, U. W. Skardowska (1998). Extensions and intentions in rough set theory, Information Sciences, 107: 149 - 167.

Davvar B. (2008). A short note on algebraic T-rough sets, Information Sciences, 178: 3247 - 3252.

Du W., B. Hu and J. Zhao (2012). On (⊥, ⊤) - Generalized Fuzzy Rough Sets, Fuzzy Information and Engineering, (3): 249 - 259.

Dubois D., H Prade (1990). Rough fuzzy sets and fuzzy rough sets, International Journal of General Systems, 17: 191 - 209.

Hu Q. H., D. R. Yu, Z. X. Xie (2006). Information-preserving hybrid data reduction based on fuzzy-rough techniques, Pattern Recognition Letters, 27(5): 414 - 423.

Kryszkiewicz M. (1999). Rules in incomplete information systems, Information Sciences, 113: 271 - 292.

Liu W., J. Yao and Y. Yao (2004). Rough approximation under level fuzzy sets, Rough set and current trend in computing (3066), pp. 78 - 83.

Lowen R. (1976). Fuzzy topological spaces and fuzzy compactness, Journal of mathematical analysis and application, 56: 621 - 633.

Pawlak Z. (1982). Rough sets, International Journal of Computer and Information Sciences, 11(5): 341 - 356.

Podsiatlo M. and M. Rybinski (2015). Application of fuzzy rough set to financial time series forecasting, Pattern recognition and machine intelligence (9124), pp. 397 - 406.

Tan A. and J. Li (2015). A kind of approximations of generalized rough set model, International journal of Machine learning and cybernetic, 6(3): 455 - 463.

Wu W. Z., J. S. Mi, W. X. Zhang (2003). Generalized fuzzy rough sets, Information Sciences, 151: 263 - 282.

Wu W. Z., Y. H. Xu (2013). On fuzzy topological structures of rough fuzzy sets, Transactions on rough sets XVI, LNCS 7736, Springer - Verlag Berlin Heidelberg, pp. 125 - 143.

Yao Y. Y. (1997), Combination of rough and fuzzy sets based on α- level sets, Rough sets and Data mining: analysis for imprecise data, Lin, T.Y. and Cercone, N. (Eds.), Kluwer Academic Publisher, Boston, pp. 301 - 321.

Yao Y. Y. (1998), Constructive and algebraic methods of the theory of rough sets, Information Sciences 109, pp. 21 - 47.

Yu H. and W. Zhan (2014). On the topologies properties of generalized rough sets, Information Sciences 263, pp. 141-152.

Zadeh L. A. (1965), Fuzzy Sets, Information and Control, 8(3): 338 - 353.

Zhao X., J. Xu and J. Gao (2009), An extension of Rough fuzzy set on the fuzzy approximation space, Fuzzy Information and Engineer, AISC 62, © Springer- Verlag Berlin Heidelberg, pp. 621 - 627.